

Ideal and nonideal electromagnetic cloaks

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We employ the analytical results for the spatial transformation of the electromagnetic fields to obtain and analyze explicit expressions for the structure of the electromagnetic fields in invisibility cloaks, beam splitters, and field concentrators. We study the efficiency of nonideal electromagnetic cloaks and discuss the effect of scattering losses on the cloak invisibility.

Cloaking of objects was first explicitly suggested by Pendry [1] as a method to cover an object by a layer of a composite material with varying characteristics in such a way that the scattering of electromagnetic waves from this object vanishes, so that the structure of the electromagnetic field outside the cloak appears as if the object and cloak are absent. The required material for creating such invisibility cloaks should have rather complex anisotropic, spatially varying properties. Moreover, such a material should possess both nontrivial dielectric and magnetic responses. Generally speaking, it is impossible to find dielectric media with the required varying properties in nature, and therefore the first electromagnetic cloak realized experimentally employed microwave metamaterials [2]. Metamaterials are artificial microstructured materials with electric and magnetic properties designed by careful engineering of their constituents. The constituents of these composite materials are normally sub-wavelength electric and magnetic resonators, with the specific properties determined by their geometry. The metamaterials can also be anisotropic, and thus they can satisfy, in principle, all the requirements for realizing invisibility cloaks for electromagnetic waves.

The concept of the electromagnetic cloaking is based on the coordinate transformation of space and the corresponding transformation of Maxwell's equations, which provide the required expressions for the effective spatially varying dielectric permittivity and magnetic permeability of the cloak medium [1]. The possibility of creating such cloaks has been confirmed experimentally [2], and it was further discussed and demonstrated in many numerical studies [3, 4, 5, 6, 7, 8, 9, 10]. Numerical simulations are usually performed either by using commercial finite-element equation solvers [1, 7, 8, 9, 10] or by employing the decomposition of the electromagnetic fields into a set of the corresponding eigenmodes [3, 4, 5, 6]. Several studies have analyzed a possibility of creating simplified cloaks [7, 9], since the required media parameters for an ideal cloaking are extremely complex for their realization in experiment. In particular, an optimization of the cloaking parameters for suppressing scattering has been analyzed in Refs. [5, 9]. In this context, it is worth mentioning an earlier pioneering study [11], where an analogy of the coordinate transformation to an effective magneto-dielectric medium was discussed for some

particular cases.

In this Letter we analyze the explicit analytical expressions for the structure of the effective electromagnetic fields caused by the spatially varying coordinate transformations. In a sharp contrast to the rather formal transformations of Maxwell's equations in the Minkovskii form [12] used in the original pioneering paper [1] and all subsequent publications, we present a simpler form of the time-independent transformations that can be immediately employed for solving many important problems. Based on these results we suggest a simple recipe for evaluating the screening efficiency of non-ideal cloaks and calculate their scattering dipole moment.

General formalism. We start our analysis by considering the propagation of monochromatic (i.e. $\sim \exp(i\omega t)$) electromagnetic fields described by Maxwell's equations

$$\text{curl} \mathbf{H} = \frac{i\omega}{c} \mathbf{D}, \text{curl} \mathbf{E} = -\frac{i\omega}{c} \mathbf{B}. \quad (1)$$

First, Eq. (1) can be written in the component notations in the following form,

$$E^{abc} \mathbf{x}_a \frac{\partial}{\partial x^b} H_c = \frac{i\omega}{c} \mathbf{x}_a D^a, \quad (2)$$

where E^{abc} is the totally antisymmetric symbol [13], \mathbf{x}_a is the orthonormal basis corresponding to the Cartesian coordinates, the indices (a, b, c) run through the values $(1, 2, 3)$, and x^a are the Cartesian coordinates corresponding to the orthonormal basis \mathbf{x}_a . In a medium with the dielectric permittivity $\hat{\epsilon}_0$, the electric displacement vector \mathbf{D} can be expressed in terms of the electric field as $D^a = \hat{\epsilon}_0^{ab} E_b$. The magnetic displacement vector \mathbf{B} can be expressed as $B^a = \hat{\mu}_0^{ab} H_b$, in a medium with magnetic permeability $\hat{\mu}_0$. We note that the repeated upper and lower indices imply summation (Einstein convention).

Now we introduce new coordinates \mathbf{y} through the transformations $\mathbf{y} = \mathbf{y}(\mathbf{x})$ with the metric tensor g^{ij} defined in a standard way,

$$g^{ij}(x) = \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} \delta^{ab}, \quad (3)$$

where δ^{ab} is Kronecker's delta-function. The determinant $g(x) = \det ||g^{ij}(x)||$ is of a particular importance for the coordinate transformations.

In the framework of these new coordinates the l.h.s. of Eq. (2) can be re-written in the form,

$$\frac{1}{2}E^{abc}\left(\frac{\partial y^i}{\partial x^b}\frac{\partial H_c}{\partial y^i}-\frac{\partial y^j}{\partial x^c}\frac{\partial H_b}{\partial y^j}\right)=\frac{1}{2}E^{abc}\left(\frac{\partial y^i}{\partial x^b}\frac{\partial y^k}{\partial x^c}\frac{\partial \tilde{H}_k}{\partial y^i}-\frac{\partial y^k}{\partial x^c}\frac{\partial y^i}{\partial x^b}\frac{\partial \tilde{H}_i}{\partial y^k}\right), \quad (4)$$

where $\tilde{\mathbf{H}}$ is the transformed magnetic field defined as

$$\tilde{H}_k = \frac{\partial x^c}{\partial y^k} H_c, \quad H_c = \frac{\partial y^k}{\partial x^c} \tilde{H}_k. \quad (5)$$

The totally antisymmetric symbol is a tensor density, and real tensor (Levi-Civita tensor, ϵ^{ijk}), which coincides with E^{ijk} in Cartesian coordinates, is related to the E^{ijk} through $\epsilon^{ijk} = \sqrt{g}E^{ijk}$, and it transforms under coordinate transformation according to the tensor rules

$$\epsilon^{lik} = \frac{\partial y^l}{\partial x^a} \frac{\partial y^i}{\partial x^b} \frac{\partial y^k}{\partial x^c} E^{abc} = \sqrt{g}E^{lik}. \quad (6)$$

Multiplying both parts of Eq. (2) by $\partial y^l/\partial x^a$ and using Eq. (6), we obtain

$$\frac{1}{2}\sqrt{g}\left[E^{lik}\frac{\partial \tilde{H}_k}{\partial y^i}-E^{lki}\frac{\partial \tilde{H}_i}{\partial y^k}\right]=\frac{i\omega}{c}\frac{\partial y^l}{\partial x^a}\epsilon_0^{ab}\frac{\partial y^n}{\partial x^b}\tilde{E}_n,$$

where $\tilde{\mathbf{E}}$ is the transformed electric field defined as

$$\tilde{E}_n = \frac{\partial x^b}{\partial y^n} E_b, \quad E_b = \frac{\partial y^n}{\partial x^b} \tilde{E}_n. \quad (7)$$

As a result, we arrive at the transformed equation

$$E^{lik}\frac{\partial \tilde{H}_k}{\partial y^i} = (\text{curl}\tilde{\mathbf{H}})^l = \frac{i\omega}{c}\frac{1}{\sqrt{g}}\frac{\partial y^l}{\partial x^a}\frac{\partial y^n}{\partial x^b}\epsilon_0^{ab}\tilde{E}_n, \quad (8)$$

which coincides with the original Maxwell's equations but it is written for the new electromagnetic fields in the new (spatially transformed) space. The left-hand side of this equation contains the curl operation on the transformed magnetic field, whereas in the right-hand side of Eq. (8) we have a new electric displacement vector which is defined as the product of the new dielectric tensor,

$$\epsilon_{\text{eff}}^{ln} = \frac{1}{\sqrt{g}}\frac{\partial y^l}{\partial x^a}\frac{\partial y^n}{\partial x^b}\epsilon_0^{ab}, \quad (9)$$

and the new electric field (7). Applying the same transformation procedure to the second equation of Eq. (1), we obtain the corresponding expression for the effective magnetic permeability,

$$\mu_{\text{eff}}^{ln} = \frac{1}{\sqrt{g}}\frac{\partial y^l}{\partial x^a}\frac{\partial y^n}{\partial x^b}\mu_0^{ab}. \quad (10)$$

When the initial medium represents vacuum ($\epsilon_0^{ab} = \mu_0^{ab} = \delta^{ab}$), the resulting new dielectric tensor coincides with the effective magnetic tensor.

Our results for the effective material parameters coincide with those of Ref. [14] but, in addition, they also provide *explicit analytical formulas* for the transformed fields. Thus, we can find analytically the field distribution for any specific cloak *without* solving Maxwell's equations in the transformed geometry, but just transforming the free-space fields. We have both the expressions for the structure of the cloak and equations (5) and (7) for the field transformations in the cloak. As a straightforward application, our results allow analytical estimations of the scattering from non-ideal cloaks, and thus a potential optimization of their parameters.

Examples. We use those results for calculating the field structure in both cylindrical and spherical cloaks, field concentrators, and other devices designed via the transformation optics. As the first example, we consider the linear coordinate transformation for the cylindrical cloak [14], $X = X(x, y)$, $Y = Y(x, y)$, $Z = z$, where the radius is transformed as $R = a + r(b - a)/b$, where $R = \sqrt{X^2 + Y^2}$, $r = \sqrt{x^2 + y^2}$. The cloak occupies the space $a < R < b$, where a and b are the inner and outer radii of the cloak, respectively. To find the field distribution in the cloak illuminated by a plane electromagnetic wave, we write the plane wave in vacuum as $H_x = \exp(i\kappa_y y)$, $H_y = 0$, $E_z = \exp(i\kappa_y y)$. Applying now the transformations according to Eqs. (5), we find the field distribution in the electromagnetic cloak with the material parameters (9) and (10), $\tilde{E}_z = \exp(i\kappa_y Y)[b(b - a)^{-1}(1 - aR^{-1})]$, $\tilde{H}_x = b(b - a)^{-1}(1 - aY^2R^{-3})\tilde{E}_z$, $\tilde{H}_y = abXY(b - a)^{-1}R^{-3}\tilde{E}_z$, for $a < R < b$. The field vanishes inside the cloaked area, whereas outside of the cloak the field is an unperturbed plane wave. At the external surface of the cloak (at $R = b$), the normal component of the magnetic field is discontinuous, since the linear space transformation produces discontinuities of the cloak material parameters. Using the transformation function

$$r = b - a \left(\frac{b - R}{b - a}\right)^\beta \quad (11)$$

provides, for large enough values of β ($\beta > 1$) continuity of the field components at the external surface of the cloak. As a matter of fact, Eqs. (5) and (7) allow us to find the field structure for *an arbitrary continuous space transformation*. Though for an arbitrary transformation one cannot find the analytical expressions for the electromagnetic fields, simple numerical calculations provide the required results without solving Maxwell's equations.

As an example of a cloak of complex shape, we consider cut-type space transformation where the cloak covers an arbitrary area 'cut' from the space (see, e.g., Fig. 1). We define the internal surface of the cloak as a set of all points which are closer than b to the cut line $X_{\text{cut}}(\xi), Y_{\text{cut}}(\xi)$. For each point (X, Y) we find the corresponding closest point on the cut line $(X_{\text{cut}}^0, Y_{\text{cut}}^0)(X, Y)$ by finding the minima of the function $[(X - X_{\text{cut}}(\xi))^2 + (Y - Y_{\text{cut}}(\xi))^2]^{1/2}$ for all possible ξ . We define the distance from the 'cut' as $R(X, Y) =$

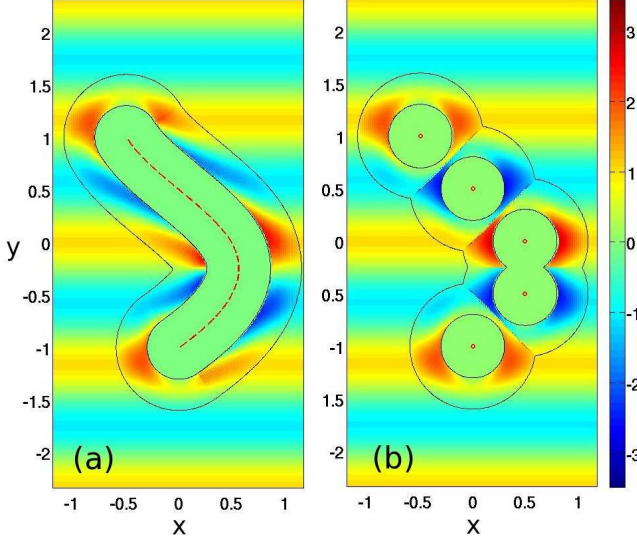


FIG. 1: (color online) Structure of the magnetic field H_x for (a) an arbitrary-shaped cloak and (b) multiple cloaks produced by the space transformation in accord with Eq. (11), where we take the parameters $a = 0.3$, $b = 0.6$, and $\beta = 2$. The cloak is illuminated by a plane wave, $H_x^0 = \exp(i\kappa_y y)$ for $\kappa_y = 5.4$. Center of the cloak is shown by red line, in the case (a), and dots, in the case (b).

$[(X - X_{\text{cut}}^0)^2 + (Y - Y_{\text{cut}}^0)^2]^{1/2}$ and create the cloaking area $a < R < b$ by transforming the space according to Eq. (11). In the area of the cloak, $a < R < b$, we define the new coordinates (x, y) as

$$x = X_{\text{cut}}^0 + (X - X_{\text{cut}}^0) \frac{r}{R}, \quad y = Y_{\text{cut}}^0 + (Y - Y_{\text{cut}}^0) \frac{r}{R},$$

which provide a conformal mapping of (x, y) into (X, Y) .

Figure 1(a) shows the distribution of the x -component of the magnetic field for the plane wave incident on the cloak. The cut is created through the graphic input in Matlab, where we specify several points (five points, in this example), which are connected by a smooth line through the cubic spline function. Figure 1(b) shows the field distribution in the cloaking problem with several cylindrical cloaks centered at the same five points.

Parameters of transformation $r(R)$ are $a = 0.3$, $b = 0.6$, $\beta = 2$. The field distribution is calculated for the incident plane wave $H_x = \exp(i\kappa_y y)$ with $\kappa_y = 5.4$.

We note that, despite the divergence of the dielectric and magnetic functions at the internal interface of the cloak, the electric and magnetic fields remain finite. For example, the radial and tangential components of the magnetic field can be found as

$$|H_R| = \frac{b}{(b-a)} \frac{|X|}{R}, \quad |H_\tau| = \frac{b}{(b-a)} \frac{|Y|}{R} \left(1 - \frac{a}{R}\right).$$

We note that $|H_R|$ is finite at the internal interface of the cloak, while $|H_\tau|$ vanishes when $R \rightarrow a$.

From the uniqueness theorem of electromagnetism we can conclude that since the tangential components of the

electric or magnetic field at the closed surface vanish, the field in the volume surrounded by this surface vanishes as well; this means that we indeed have the volume ($R < a$) concealed from the electromagnetic radiation.

Effectiveness of non-ideal cloaks. Above we demonstrated that a cloak with the parameters defined by Eqs. (9) and (10) provides complete invisibility of the objects hidden inside. Moreover, such a cloak works for an arbitrary transformation function $\mathbf{y} = \mathbf{y}(\mathbf{x})$, either smooth or piece-wise continuous [7]. However, this kind of ideal cloaking seems to be impossible in practice, since it requires infinite values of material parameters at the internal surface. The simplest way to overcome this difficulty is to truncate the transformed area. For example, for the coordinate transformation with the internal radius a we create the cloak only for $R > a + \delta a$, and then place a perfectly conducting metal surface at $R = a + \delta a$. Now, if we make an inverse transformation from the cloak to vacuum, the metallic surface transforms into a metallic cylinder with the radius r_{eff} . If $dR/dr|_{r=a} \neq 0$, then $r_{\text{eff}} \approx \delta a (dR/dr|_{r=a})^{-1}$, the simplified cloak scatters light as a metallic cylinder of the radius r_{eff} , which can be much smaller than the size of the hidden region a .

One of the main limitations of the resonant medium used for creating electromagnetic cloaks is losses. To estimate the effect of small losses on the cloak performance, we assume that $\hat{\varepsilon} = \hat{\mu} = \hat{\varepsilon}' + i\hat{\varepsilon}''$, $\hat{\varepsilon}'' \ll \hat{\varepsilon}'$, whereas $\hat{\varepsilon}'$ is given by Eq. (9). We consider a plane wave incident on such a cloak. After applying the inverse coordinate transformation, which transforms the electromagnetic fields into vacuum plane waves, we obtain in original space the excited effective electric and magnetic currents dependent on vacuum electric and magnetic fields,

$$j_{e(m)}^a = \sigma^{ab} E_b^v(H_b^v), \quad \sigma^{ab} = -\frac{\omega \sqrt{g}}{4\pi} \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \varepsilon^{ij}.$$

These currents lead to the scattering radiation which can be easily evaluated either analytically or numerically.

Next, we can estimate the effect of mismatched dielectric permittivity and magnetic permeability on the cloak performance. We assume that $\hat{\varepsilon} = \hat{\mu} + \delta\hat{\varepsilon} = \hat{\varepsilon}_0 + \delta\hat{\varepsilon}$, where $\hat{\varepsilon}_0$ corresponds to an ideal cloak and $|\delta\hat{\varepsilon}| \ll |\hat{\varepsilon}_0|$. Then, we apply the inverse transformation $\mathbf{x} \rightarrow \mathbf{y}$ and find that in the original space the dielectric permittivity differs from the unity by the value

$$\delta\hat{\varepsilon}^{ab} = \sqrt{g} \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \delta\varepsilon^{ij}.$$

If we assume that $(\hat{\varepsilon} - \hat{\mu}) \propto \hat{\varepsilon}$, namely $\delta\varepsilon^{ij} = \alpha \varepsilon_0^{ij}$ (for $\alpha \ll 1$), then $\delta\hat{\varepsilon}^{ab} = \alpha \delta^{ab}$. Thus, in this case the non-ideal cloak scatters the waves as an object (cylinder of the radius b for the cylindrical cloak) with a scalar dielectric permittivity $1 + \alpha \approx 1$.

Now we consider $\delta\varepsilon^{ij} = \alpha \delta^{ij}$, for $\alpha \ll 1$. This gives us a relatively simple result for $\delta\hat{\varepsilon}^{ab}$,

$$\delta\hat{\varepsilon}^{ab} = \alpha \sqrt{g} \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \delta^{ij} \quad (12)$$

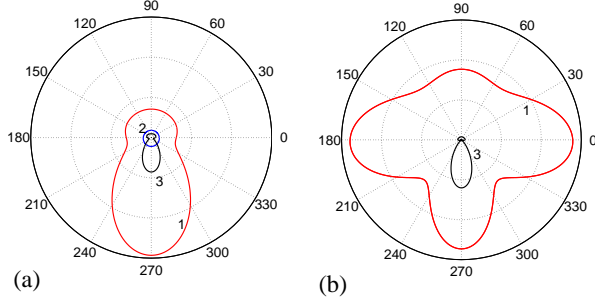


FIG. 2: (Color online) Scattering patterns for non-ideal cloak for TE (a) and TM (b) waves. Red lines (1) indicate scattering from bare metallic cylinder, while blue line (2) corresponds to the scattering from the cloaked cylinder with truncation level $r_0 = 0.05b$ and $\alpha = 0$. Black curves (3) correspond to the scattering (mainly forward) on the dielectric cloak with parameter $\alpha = 0.1$ for the figure (a) and $\alpha = 0.02$ for the figure (b).

and also an expression for the electric current

$$\mathbf{j}_{\text{eff}} \approx \frac{i\omega}{4\pi} \alpha \hat{\varepsilon}^{-1} \mathbf{E}^v, \quad (13)$$

which can be used for calculating the scattered fields. It is worth mentioning that in this case the cloak is sensitive to small perturbations of the parameters. Indeed, the inverse dielectric permittivity has a singularity at the inner surface of the cloak, so if the truncation value δa is small then the parameter α has to be very small as well so that it does not perturb the structure of the plane wave. This result has been predicted earlier in Ref. [4]. We also note that very similar expressions for the electric and magnetic currents in the lossy case may also lead (for particular forms of $\hat{\varepsilon}''$) to such a strong sensitivity.

As an example let us consider plane wave scattering on a slightly non-ideal cloak with the magnetic permeability tensor components $\mu_{rr} = (R - a)/R$, $\mu_{\phi\phi} = R/(R - a)$,

$\mu_{zz} = \gamma^{-2}(R - a)/R$, $\gamma = (b - a)/b$ and dielectric permittivity tensor $\varepsilon_{ii} = \mu_{ii} + \alpha$, $\alpha \ll 1$. We suppose that the wave propagates in y -direction, its wavenumber, $k_y = 5.4$, and inner and outer radii of the cloak are $a = 0.3$, $b = 0.6$, respectively. For the case of $\alpha = 0$ such cloak is ideal, and it corresponds to the linear coordinate transformation $R = a + \gamma r$, $0 < r < b$. In order to avoid the singularity we place a metallic cylinder at $R = a + \delta a$, which conceals the inner region of the cloak but it also distorts scattering performance. Scattered fields are regarded as weak, and we use the perturbation approach to calculate them. Let us make an inverse coordinate transformation $R \rightarrow r$, so that the metal screen radius transforms to $r_0 = \delta a/\gamma$. We calculate the effective current according to the Eq. (13) and find the scattering field pattern for both TE ($\mathbf{E}^0 = \mathbf{z}^0 E$) and TM ($\mathbf{H}^0 = \mathbf{z}^0 H$) waves (see Figures 2a,b). One can see that for the TM polarization the scattering from non-ideal cloak is negligible, and in the shown scale it is represented by a point in the center of the Fig. (2)(b). If the truncation level, r_0 , is reduced, the scattered radiation for TM wave is also reduced. On the contrary, TE wave scattering is significantly enhanced, e.g., if we decrease r_0 to $0.005b$, then the scattered power becomes four times larger.

This approach allows to estimate the scattering losses in an arbitrary nonideal cloak (e.g., such as that shown in Fig. 1), since the effective currents can be calculated using Eq. (13) with the plane wave excitation.

In conclusion, we have employed the analytical results for the distribution of electromagnetic fields in the transformed coordinates to analyze an explicit solution for the problem of the wave scattering in electromagnetic cloaks, beam splitters, energy concentrators, and other devices of the transformation optics. Employing those results, we have derived a simple criterion of the efficiency of nonideal cloaks based on estimations of wave scattering.

The authors thank Prof. V.E. Semenov for valuable discussions. This work was supported by the Australian Research Council and the Russian Fund for Basic Research (grant No. 08-02-00379).

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